1 MPHY-CC-6 Electrodynamics & Plasma physics unit-1

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MAXWELL'S EQUATION OF ELECTROMAGNETIC WAVES

1. Maxwell's first equation: $\vec{\nabla}\cdot\vec{D} = \rho$ or $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

It is Gauss's law of electrostatics.

Where $D = \varepsilon_0 E$, D = Electric displacement or Electric flux density, E = Electric field intensity, ρ = volume charge density. $\varepsilon_0 \to$ permittivity of free space.

Integral form: $\vec{\nabla}\cdot\vec{D} = \rho$

Integrating over volume

$$\int_V \vec{\nabla}\cdot\vec{D}\, dv = \int_V \rho\, dv$$

But from Gauss divergence theorem, $\int_V \vec{\nabla}\cdot\vec{D}\, dv = \oint_S \vec{D}\cdot d\vec{s}$

so $$\boxed{\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dv = q \text{ or } \oint \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\int_V \rho dv = \frac{q}{\varepsilon_0}}$$

It is Maxwell's first equation in 'integral form.

2. Maxwell's second equation: $\vec{\nabla}\cdot\vec{B} = 0$ or $\vec{\nabla}\cdot\vec{H} = 0$

It is Gauss's law of magnetism (Magnetostatics)

Where $B = \mu_0 H$, B = magnetic field or magnetic flux density H = magnetic field intensity, μ_0 = permeability of free space.

Integral form: $\vec{\nabla}\cdot\vec{B} = 0$

Integrating over volume

$$\int_V \vec{\nabla}\cdot\vec{B}\, dv = 0$$

But from Gauss divergence theorem, $\int_V \vec{\nabla}\cdot\vec{B}\, dV = \oint_S \vec{B}\cdot d\vec{s}$

so $\boxed{\oint_S \vec{B}\cdot d\vec{s} = 0}$ or $\oint_S \mu_0 \vec{H}\cdot d\vec{s} = 0 \quad \because \vec{B} = \mu_0\vec{H}$

$$\Rightarrow \oint_S \vec{H}\cdot d\vec{s} = 0$$

Thus $$\boxed{\oint_S \vec{B}\cdot d\vec{s} = 0 \quad \text{or} \quad \oint \vec{H}\cdot d\vec{s} = 0}$$

It is Maxwell's second equation in integral form.

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3. Maxwell's third equation: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

It is Faraday's law of electromagnetic induction.

Integral form: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Integrating both sides over surface S

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

But from Stoke's theorem, $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = \oint_L \vec{E}\cdot d\vec{l}$

so $$\boxed{\oint_L \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{s} \Rightarrow \oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}}$$

It is Maxwell's third equation in integral form.

4. Maxwell's fourth equation: $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

It is modified Ampere's circuital law

where J = current density, $\frac{\partial \vec{D}}{\partial t}$ = Displacement current density.

Integral form: $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Integrating both sides over surface S

$$\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{s} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s}$$

But from Stoke's theorem, $\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{s} = \oint \vec{H}\cdot d\vec{l}$

so $$\boxed{\oint \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s} \text{ or } \oint \vec{B}\cdot d\vec{l} = \mu_0 \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s}}$$

It is Maxwell's four equation in integral form.

Thus Maxwell's equation of electromagnetism is

In differential form	In integral form
1. $\vec{\nabla}\cdot\vec{D} = \rho$ or $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$	1. $\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dv$ or $\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\int_V \rho\, dv$
2. $\vec{\nabla}\cdot\vec{B} = 0$ or $\vec{\nabla}\cdot\vec{H} = 0$	2. $\oint_S \vec{B}\cdot d\vec{s} = 0$ or $\oint_S \vec{H}\cdot d\vec{s} = 0$
3. $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	3. $\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$
4. $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	4. $\oint \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s}$ or $\oint \vec{B}\cdot d\vec{l} = \mu_0 \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s}$

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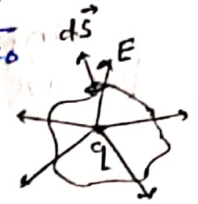
Derivation of Maxwell's equation of Electromagnetism.

Maxwell's first equation: $\vec{\nabla}\cdot\vec{D} = \rho$ or $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$



From Gauss's theorem of electrostatics, total electric flux through any closed surface is equal to total charge enclosed by the closed surface divided by ε_0.

i.e $\oint_S \vec{E}\cdot d\vec{s} = \frac{q}{\varepsilon_0}$

Now volume charge density $\rho = \frac{dq}{dv} \Rightarrow dq = \rho\, dv$

$\Rightarrow q = \int_V \rho\, dv$

Thus $\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\int_V \rho\, dv$

But from Gauss divergence theorem,

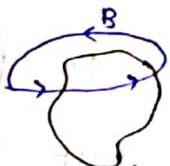
$\int_V (\vec{\nabla}\cdot\vec{E})\, dv = \oint_S \vec{E}\cdot d\vec{s}$

So $\int_V (\vec{\nabla}\cdot\vec{E})\, dv = \frac{1}{\varepsilon_0}\int_V \rho\, dv \Rightarrow \int_V \left(\vec{\nabla}\cdot\vec{E} - \frac{\rho}{\varepsilon_0}\right) dv = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{E} - \rho/\varepsilon_0 = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0}$ or $\boxed{\vec{\nabla}\cdot\vec{D} = \rho}$

$\because \vec{D} = \varepsilon_0\vec{E}$

Maxwell's second equation: $\vec{\nabla}\cdot\vec{B} = 0$ or $\vec{\nabla}\cdot\vec{H} = 0$



From Gauss's theorem of magnetism, total magnetic flux through any closed surface is equal to zero because magnetic field line is a closed continuous loop so total magnetic field lines entering a closed surface is equal to total magnetic field lines leaving the closed surface.

i.e, $\oint \vec{B}\cdot d\vec{s} = 0$

Using Gauss's divergence theorem, $\int_V (\vec{\nabla}\cdot\vec{B})\, dv = \oint \vec{B}\cdot d\vec{s}$

so $\int_V (\vec{\nabla}\cdot\vec{B})\, dv = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{H} = 0}$

$\because B = \mu_0 H$

Maxwell's third equation: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

From Faraday's law of electromagnetic induction, the

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induced emf produced in a closed loop is equal to negative of time rate of change in magnetic flux linked with the closed loop.

i.e, $e = -\frac{d\phi}{dt}$

Emf (e) is defined as the work done in carrying unit positive charge (+1C) round a closed circuit.

so $e = \oint \vec{E}\cdot d\vec{l} \Rightarrow \oint \vec{E}\cdot d\vec{l} = -\frac{\partial \phi}{\partial t}$

But from Stoke's theorem, $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = \oint \vec{E}\cdot d\vec{l}$

so $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\frac{\partial\phi}{\partial t}$ but $\phi = \int_S \vec{B}\cdot d\vec{s}$

so $\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{s} \Rightarrow \int_S \left(\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{s} = 0$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Maxwell's fourth equation: $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ concept of displacement current.

From Ampere's circuital law, line integration of magnetic field along any closed loop is equal to μ_0 times of total current passing through the surface bounded by the closed loop. ie, $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$

but $I = \oint_S \vec{J}\cdot d\vec{s}$ and $\vec{B} = \mu_0 H$

so $\oint \mu_0 \vec{H}\cdot d\vec{l} = \mu_0 \oint_S \vec{J}\cdot d\vec{s} \Rightarrow \oint \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s}$

But from Stoke's theorem, $\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{s} = \oint \vec{H}\cdot d\vec{l}$

so $\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{s} = \int_S \vec{J}\cdot d\vec{s} \Rightarrow \int (\vec{\nabla}\times\vec{H} - \vec{J})\cdot d\vec{s} = 0$

$$\Rightarrow \vec{\nabla}\times\vec{H} - \vec{J} = 0 \Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}} \quad \text{---} (1)$$

The relation (1) stands only for steady closed current.

Now $\text{div}(\text{curl}\,\vec{H}) = 0$ because divergen of curl of any vector is zero.

$\Rightarrow \text{div}\,\vec{J} = 0$ --- (2) $\because \vec{\nabla}\times\vec{H} = \vec{J}$.

From equation of continuity, $\text{div}\, J = -\frac{\delta \rho}{\delta t}$ —— (3)

From equⁿˢ (2) and (3), $\frac{\delta \rho}{\delta t} = 0 \Rightarrow \rho = \text{constant}$

Therefore equⁿ (1) is valid only for constant volume charge density. If volume charge density ρ be variable then $\text{div}\, J \neq 0$

Maxwell suggested that equⁿ (1) is incomplete when volume charge density ρ is variable. For variable volume charge density ρ in space i.e, for varying field in space, Maxwell suggested that use $J + J_d$ in place of J and this J_d is known as displacement current density.

Hence equⁿ (1) will become $\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_d$ —— (4)

Since $\text{div}(\vec{\nabla} \times \vec{H}) = 0$ so $\text{div}(\vec{J} + \vec{J}_d) = 0$

$\Rightarrow \text{div}\, \vec{J} + \text{div}\, \vec{J}_d = 0 \Rightarrow \text{div}\, \vec{J}_d = -\text{div}\, \vec{J}$

$\Rightarrow \text{div}\, \vec{J}_d = -\left(-\frac{\delta \rho}{\delta t}\right) \quad \because \text{div}\, J = -\frac{\delta \rho}{\delta t}$ from equⁿ (3)

$\Rightarrow \text{div}\, \vec{J}_d = \frac{\delta \rho}{\delta t} \Rightarrow \vec{\nabla} \cdot \vec{J}_d = \frac{\delta}{\delta t}(\vec{\nabla} \cdot \vec{D})$ from Maxwell's first law $\rho = \vec{\nabla} \cdot \vec{D}$

$\Rightarrow \vec{\nabla} \cdot \vec{J}_d = \vec{\nabla} \cdot \frac{\delta \vec{D}}{\delta t} \Rightarrow \vec{\nabla}\left(\vec{J}_d - \frac{\delta \vec{D}}{dt}\right) = 0$

$\Rightarrow \vec{J}_d - \frac{\delta \vec{D}}{\delta t} = 0 \Rightarrow \boxed{\vec{J}_d = \frac{\delta \vec{D}}{\delta t}}$ It is formula for displacement current density.

put in equⁿ (4), we get

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t}} \text{ or } \boxed{\vec{\nabla} \times \vec{B} = \mu_0\left(\vec{J} + \frac{\delta \vec{D}}{\delta t}\right)}$$

Displacement current $\boxed{I_d = \int_s \vec{J}_d \cdot d\vec{S}}$